INFORMATION AND CONTROL 19, 30–40 (1971)

A Categorical Characterization of General Automata

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Let $\mathscr{A}'$ be a category of input objects and $\mathscr{A}$ a category of output objects. Assume that $\mathscr{A}'$ has limits and $\mathscr{A}$ has colimits; then, the behavior of every diagram in the category of abstract machines $(\mathscr{A}', \mathscr{A})$ over a connected scheme can be recorded in its limit. Thus, the theory of limit preserving functors describes the interconnections between the various traits that the total system exhibits. Hence, "the systems theorist is told" what jobs an arbitrary system can do for the given system.

I. INTRODUCTION

Systems engineers (Mesarovic, 1968) have found a model of a general system consisting of a subset of the cartesian product of a finite number of coordinate objects to be of general descriptive power. We begin with that notion here.

General System

Let $N = \{1,..., n\}$ and $\mathscr{V} : V_1 ,..., V_n$ be coordinate sets (thus, all variables of the system are represented in these n symbols). As is very often the case, the members of each V_i can be constant elements or other sets. Let $\mathscr{V} = \mathscr{V}_x \cup \mathscr{V}_y = \{V_i : i \in I_x\} \cup \{V_j : j \in I_y\}$, and $I_x \cup I_y = N$.

V_1^x	V_1^y	$\cdot \quad \cdot \quad \cdot$	$V_{n_1}^x$	$V_{n_2}^y$

$n_1 + n_2 = n$

Arbitrarily, call $\mathscr{V}_x$ *input objects* and $\mathscr{V}_y$ *output objects*. Let $A \in \mathscr{V}_x$, $B \in \mathscr{V}_y$, and $R \subseteq A \times B$ a relation. Let $R_1 \subseteq A_1 \times B_1$, $R_2 \subseteq A_2 \times B_2$, where $B_1 = A_2$. Then, define (this is consistent with Freyd (1964) and Lawvere (1965))

$$R_2 \circ R_1 \subseteq A_1 \times B_2 = \{(a_1 , b_2) : \exists c \in B_1 = A_2 \ni (a_1 , c) \in R_1 , (c, b_2) \in R_2\}.$$

Composition "$\circ$" is called *relative product.* Let $A_1, A_2 \in \mathscr{V}_x$. Let $[A_1, A_2] \subseteq \{f : f \text{ is a } \textit{function} \text{ from } A_1 \text{ to } A_2\}$. Let $h_x \in [A_1, A_2]$ be called an *incoding morphism.* Similarly, for every $B_1, B_2 \in \mathscr{V}_y$ let $[B_1, B_2]_y \subseteq \{g : g \text{ is a } \textit{function} \text{ from } B_1 \text{ to } B_2\}$. Let $h_y \in [B_1, B_2]$ be called an *outcoding morphism.*

As an example, let (Birkhoff and Bartee, 1970) A_1 be a set of message words where h_x encodes A_1 into binary form A_2.

Let $R_1, R_2 \in \mathscr{R} = \{R : \exists A \in \mathscr{V}_x, B \in \mathscr{V}_y \ni R \subseteq A \times B\}$. Suppose that there exist $h_x \in [A_1, A_2]_x$, $h_y \in [B_2, B_1]_y$ so that $R_1 = h_y \circ R_2 \circ h_x$, i.e.,

$$\begin{tikzcd}
A_2 \arrow[r, dash, "R_2"] & B_2 \arrow[d, "h_y"] \\
A_1 \arrow[u, "h_x"] \arrow[r, dash, "R_1"'] \arrow[ur, phantom, "\bullet"] & B_1
\end{tikzcd}$$

FIG. 1. A morphism between two relations.

For every $a_1 \in \text{domain } R_1$, $h_x(a_1) \in \text{domain } R_2$, there exists $b_2 \in \text{codomain } R_2$ such that $(h_x(a_1), b_2) \in R_2$, and $(a_1, h_y(b_2)) \in R_1$. Call the pair (h_x, h_y) a *morphism from* R_1 *to* R_2, denoted $R_1 \xrightarrow{(h_x, h_y)} R_2$. Assume that for every $A \in \mathscr{V}_x 1_A \in [A, A]_x$ and for every $B \in \mathscr{V}_y 1_B \in [B, B]_y$; then, $R_1 \xrightarrow{(1_A, 1_B)} R_1$. It is apparent that h_x, h_y must be functions.

Now, $\mathscr{V}_x \cup (\bigcup [A_1, A_2])$ may not yet be a category since $h_x^1 \in [A_1, A_2]$, $h_x^2 \in [A_2, A_3]$

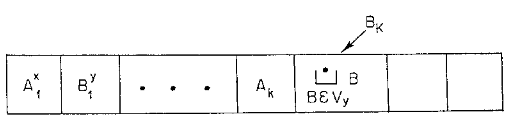
$$A_1 \xrightarrow{h_x^1} A_2 \xrightarrow{h_x^2} A_3$$

does not yet imply that the function $h_x^2 \bullet h_x^1 \in [A_1, A_3]$; hence, we must introduce a mapping $\bullet : [A_2, A_3] \times [A_1, A_2] \to [A_1, A_3]$ defined for every A_1, A_2, A_3.

Let $\mathscr{V}_x \cup (\bigcup [A_1, A_2])_\bullet$, $\mathscr{V}_y \cup (\bigcup [B_1, B_2])_\circledcirc$ be categories.

Remark 1.1. If $\text{morph}\, \mathscr{C} = \bigcup_{C \times D \in \mathscr{C} \times \mathscr{C}} [C, D]$ is not a disjoint union, replace $f \in [C, D]$ by a triple (C, f, D).

There are several problems that we next wish to discuss. First, $\dot{\bigcup}_{B \in \mathscr{V}_y} B$, the set theoretic *colimit* of the B's, is not itself in $\mathscr{V}_y$. Neither are the inclusion maps $B \overset{i}{\rightarrowtail} \dot{\bigcup}_{B \in \mathscr{V}_y} B$ morphisms of the category $\mathscr{V}_y \cup (\bigcup [B_1, B_2])_\circledcirc$. Thus, no $B \in \mathscr{V}_y$ can be replaced by $\dot{\bigcup}_{B \in \mathscr{V}_y} B$.



The notion of (out) *replacement* is fundamental in systems theory.

The second problem is that the *product* $A_1 \times \cdots \times A_k \times \cdots \times A_{n_1}$ is not in $\mathscr{V}_x$; neither are the projection maps $A_1 \times \cdots \times A_{n_1} \xrightarrow{p} A$ morphisms of the category $\mathscr{V}^x \cup (\bigcup [A_1, A_2])_\bullet$.

PROPOSITION 1.2. *Let $\mathscr{C}$ be a category with products and finite intersections, and let $\mathscr{D}$ be a diagram in $\mathscr{C}$ over a scheme (I, M, d). Then a limit for $\mathscr{D}$ is given by the family of compositions*

$$\bigcap_{m \in M} \mathrm{Equ}(p_k, \mathscr{D}(m)\, p_j) \subset \bigtimes_{h \in I} D_h \xrightarrow{p_i} D_i,$$

where $d(m) = (j, k)$, and p_i represents the i-th projection from the product. (See figures 2, 3).

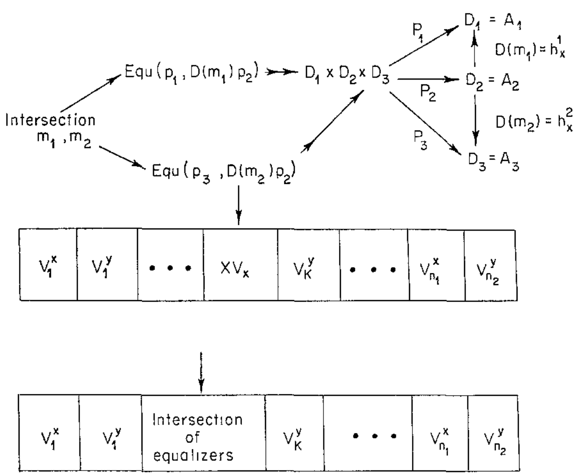


FIG. 2 AND 3. Formation of the limit in Proposition 1.2.

Thus, there are two notions of (in)*replacement* to be considered here. One allows an n_1-tuple $(v_1, \ldots, v_{n_1})$ of values in $V_1{}^x \times \cdots \times V^x_{n_1}$ to be a replacement value, and the other one allows only *some* of these n_1-tuples to be placed as input values into the system in the beginning.

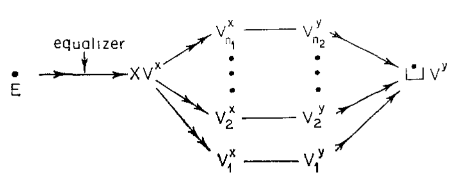


FIG. 4. The equalizer mapping into a product of input objects.

Thus, it is the function of the equalizer map to attempt to (if possible) coordinate the inputs being fed into the system.

As we shall see in Theorem 1.6 this coordination is assured only if the *diagram of relations* is *connected* in the categorical sense (Mitchell, 1965). Moreover, if connectedness holds, then one can consider a new relation $E \xrightarrow{\bar{R}} \cup V^y$. Now, $\mathscr{R}$ with morphisms $R_1 \xrightarrow{(h_x, h_y)} R_2$, as defined previously, can be made into a category. If $E \in \mathscr{V}^x$ and $\cup V^y \in \mathscr{V}^y$ with projections, equalizers, pullbacks in $\mathscr{V}^x$, and inclusions in $\mathscr{V}^y$, then $\bar{R} \in \mathscr{R}$. $\bar{R}$ is the *limit* (up to isomorphism) of the diagram in category $\mathscr{R}$. And, $\bar{R}$ carries with it the behavior (appearances, experiments) of this diagram of relations.

PROPOSITION 1.3 (Mesarovic, 1968). *Let $S \subseteq X \times Y$ be an arbitrary binary relation. There exists a family of functions $\bar{f} = \{f : X \to Y\}$ such that $S = \cup \bar{f}$.*

Hence, if a system is a collection $\mathscr{R}$ of relations R, then each R is a family of functions $R = \cup \bar{f}_R$. Thus, it is sometimes helpful to consider a system as a collection of functions. Hence, $\mathscr{R}$ is a category of functions $\bar{f}$ with $\bar{f} \xrightarrow{(h_x, h_y)} \bar{g}$.

Remark 1.4. Every system S has an initial global state object Z whose elements are precisely the functions $\bar{f}$.

THEOREM 1.5. *Let $S'' \subseteq S$ be a subsystem of S. Let category $\mathscr{V}^x$ have limits and $\mathscr{V}^y$ colimits; let $S'' = \mathscr{D}(I, M, h)$ be a connected diagram (small scheme with infinite products). Then, S'' can be represented by $V^x \xrightarrow{\bar{R}} V^y$ for some $V^x \in \mathscr{V}^x$ and some $V^y \in \mathscr{V}^y$.*

A word of caution is necessary; namely, $V^x \in \mathscr{V}''^x$, $V^y \in \mathscr{V}''^y$ may not be true.

The proof of the next theorem falls within the scope of Theorem 2.1.

THEOREM 1.6. *Hypothesis*: (1) *Let S be a system, and let χ, χ^0 be (connected, small, infinite products) categories;*

(2) *every $\chi \xrightarrow{F} \mathscr{V}^x$ has a left root $\varprojlim F$;*

(3) *every $\chi^0 \xrightarrow{F^0} \mathscr{V}^y$ has a right root $\varinjlim F^0$.*

Conclusion: (1') *S can be extended to a category;*

(2') *every $(\chi, \chi^0) \xrightarrow{(F, F^0)} S$ has a left root $\varprojlim (F, F^0)$;*

(3') *in S $\varprojlim F \xrightarrow{\overleftarrow{\lim}(F, F^0)} \varinjlim F^0$.*

II. Metamathematical Machine Theory

Let $\mathscr{A} \subseteq \mathscr{A}$ be a subcategory of the category $\mathscr{A}$. An *abstract machine* is a morphism $f \in [A', A]_{\mathscr{A}}$, where $A' \in \mathrm{Ob}\, \mathscr{A}'$, $A \in \mathrm{Ob}\, \mathscr{A}$. Let $(\mathscr{A}', \mathscr{A})$ denote the class of all such abstract machines. A commutative square $f_1 = hf_2H$ defines a *morphism* between abstract machines f_1, f_2, where $H \in \mathrm{morph}\, \mathscr{A}'$, $h \in \mathrm{morph}\, \mathscr{A}$, and will be denoted $f_1 \xrightarrow{(H,h)} f_2$.

Let $\mathscr{A}$ be a category of sets with $\mathscr{A}' \subseteq \mathscr{A}$ a subcategory of $\mathscr{A}$. Given the pair (A', A), where $A' \in \mathscr{A}'$, $A \in \mathscr{A}$, and $I = |A|$ is A as a set, choose arbitrary $f_i \in [A', A]_{\mathscr{A}}$, $\forall i \in I$; and, formally define $A' \xrightarrow{\cup_{i \in I} f_i} A$ a family of maps from A' into A. Denoting $\cup_{i \in I} f_i = \delta_I$ we have a triple $\mathfrak{A} = \langle A', \delta_I, A \rangle$.

Let $A' \in \mathscr{A}'$, $A \in \mathscr{A}$, and $\cup_{\lambda \in \Lambda} f_\lambda = \delta_\Lambda$ a collection of mappings from A' to A. This forms a triple $\langle A', \delta_\Lambda, A \rangle$ which we shall call a *general automaton* with *input object* A', *state object* A, and transition δ_Λ determined by the family of *general abstract machines* f_λ (Rine, 1970).

One may think of $A' \times A \xrightarrow{\delta_I} A$ as being defined by $\delta_I(a', a) = \delta_I(a', i) = f_i(a')$, where $i \equiv a$.

We will assume that $i \neq j$ implies that $f_i \neq f_j$, which in Arbib (1969) means that $\mathfrak{A}$ is reduced; moreover, every finite (special) automaton $\mathfrak{A}$ is (state) equivalent to a reduced $\mathfrak{A}^0$ (Arbib, 1969). Let (A', A), (B', B) be two pairs, with $I = |A|$, $J = |B|$. Choose $\delta_I = \cup f_i$ and $\delta_J = \cup g_j$, forming $\mathfrak{A} = \langle A', \delta_I, A \rangle$ and $\mathscr{B} = \langle B', \delta_J, B \rangle$. We will say that $\mathscr{B}$ *weakly simulates* $\mathfrak{A}$ iff $\exists H \in [A', B']_{\mathscr{A}'}$, $h \in [B, A]_{\mathscr{A}}$, and $\hbar \in [A, B]_{\mathscr{A}}$, i.e., $I \xrightarrow{\hbar} J$ or $\hbar(i) = j$, such that $\forall i \in I f_i = hg_{\hbar(i)}H$. Hence, we have three commutative squares:

$$\begin{tikzcd} A' \arrow[r, "f_i"] \arrow[d, "H"'] \arrow[dr, phantom, "N(i) = j"] & A \\ B' \arrow[r, "g_j"] & B \arrow[u, "h"'] \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "\sqcup f_i"] \arrow[d, "H"'] & A \\ B' \arrow[r, "\sqcup g_j"] & B \arrow[u, "h"'] \end{tikzcd}$$

$$\begin{tikzcd} A' \times A \arrow[r, "\delta_I"] \arrow[d, "(H, n)"] & A \\ B' \times B \arrow[r, "\delta_J"] & B \arrow[u, "h"'] \end{tikzcd}$$

Fig. 5. A morphism between two automata.

Let us express weak simulation by $\mathfrak{A} \xrightarrow{(H, \hbar, h)} \mathscr{B}$ or $(H, \hbar, h) \in [\mathfrak{A}, \mathscr{B}]$.

Let

$$\mathscr{H} = (H, \hbar, h) \in [\mathfrak{A}, \mathscr{B}], \qquad \mathscr{K} = (K, k\!\!\!/, k) \in [\mathscr{B}, \mathscr{C}],$$

and

$$\mathscr{M} = (M, m\!\!\!/, m) \in [\mathscr{C}, \mathscr{D}];$$

then, when the appropriate compositions exist, $\mathscr{M}(\mathscr{K}\mathscr{H}) = (\mathscr{M}\mathscr{K})\mathscr{H}$; and by Rine (1970) one can extend $(\mathscr{A}', \varDelta, \mathscr{A}) = [\langle A', \delta, A\rangle] \cup [(H, \hbar, h)]$ to a category consisting of objects and morphisms respectively.

Let $\mathscr{D}$ and $\mathscr{A}$ be categories and $F : \mathscr{D} \to \mathscr{A}$ a functor. Freyd (1964) defines the *left root* (if it exists) of F to be a constant functor $L : \mathscr{D} \to \mathscr{A}$ such that for every constant functor $C : \mathscr{D} \to \mathscr{A}$ and natural transformation $C \to F\, \exists!\, C \to L$ so that $C \to L \to F = C \to F$. The *right root* of a functor $F : \mathscr{D} \to \mathscr{A}$ is a constant functor $R : \mathscr{D} \to \mathscr{A}$ along with a natural transformation $F \to R$ so that for every constant functor $C : \mathscr{D} \to \mathscr{A}$ and transformation $F \to C\, \exists!$ transformation $R \to C$ so that $F \to R \to C = F \to C$.

With a slight modification due to the state relation $\hbar$ we will move from general abstract machines to general automata by assuming that whenever $\varDelta_1(\mathscr{H}) = \varDelta_1(\mathscr{K})$, equal codomains, (Lawvere, 1966) then $\operatorname{Im} \hbar = \operatorname{Im} k\!\!\!/$; we call this condition (H).

THEOREM 2.1. *Hypothesis*: (1) *Let $\mathscr{A}' \subseteq \mathscr{A}$ be a subcategory of $\mathscr{A}$, and let χ, χ^0 be connected, small categories with infinite products*;[1]

(2) *every $\chi \xrightarrow{F} \mathscr{A}'$ has a left root $\varprojlim F$*;

(3) *every $\chi^0 \xrightarrow{F^0} \mathscr{A}$ has a right root $\varinjlim F^0$.*

Conclusion: (1′) *$(\mathscr{A}', \mathscr{A})$ can be extended to a category*;

(2′) *(χ, χ^0) is connected*;

(3′) *every $(\chi, \chi^0) \xrightarrow{(F,F^0)^2} (\mathscr{A}', \mathscr{A})$ has a left root $\varprojlim\, (F, F^0)$*;

(4′) *in $(\mathscr{A}', \mathscr{A})$ $\varprojlim F \xrightarrow{\overleftarrow{\lim}(F,F^0)} \varinjlim F^0$.*

Proof. By making an identification from χ to (χ, χ^0) it is easy to see that (χ, χ^0) is connected. With L, R left and right roots, let $L(H) = l$ constant and $R(h) = \imath$ constant, where (by identification) $H \in \mathscr{A}'$ and $h \in \mathscr{A}$.

[1] The reader might wish to replace the schemes χ, χ^0 by a single scheme χ.

[2] $(F, F^0)/\chi = F$ and $(F, F^0)/\chi^0 = F^0$.

Now $(L, R)(H, h) \in [l, r] \neq \varnothing$, but what choice do we make? Choose $(L, R)(H, h) = p = fa_i f'$ in the following diagram:

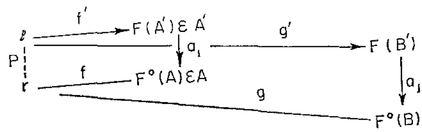


FIG. 6. The construction of the left root in $(\mathscr{A}', \mathscr{A})((\mathscr{A}', \Delta, \mathscr{A}))$.

Since $L \xrightarrow{\lambda} F$ is natural and χ, χ^0 are connected, we can complete the following two diagrams

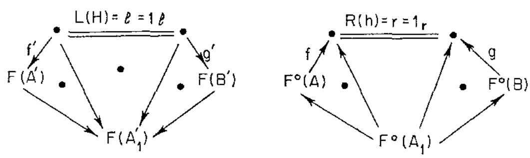


FIG. 7. The construction of the left root in $(\mathscr{A}', \mathscr{A})((\mathscr{A}', \Delta, \mathscr{A}))$.

where $p = fa_i f'$ and $p_1 = ga_j g'$. By using the same argument with (χ, χ^0) connected it follows that it makes no difference what choice we make. Thus we have these lemmas.

LEMMA. $\exists(L, R) \xrightarrow{(\lambda,\rho)} (F, F^0)$ *natural transformation.*

LEMMA. (χ, χ^0) *is connected.*

LEMMA. $p = p_1$ *and* (L, R) *is constant, independent of the path.*

Finally, we can obtain by standard argument the result (4′).

LEMMA. (L, R) *is constant,* $(\lambda, \rho) \in [(L, R), (F, F^0)]$ *is natural, and* $\forall z \in [Z, (F, F^0)]$, *where* Z *is constant,* $\exists!(\gamma', \gamma)$ *transformation resulting in a commutative diagram.*

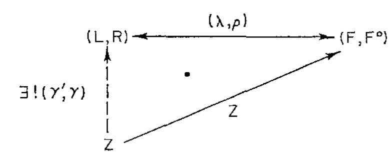


FIG. 8. The universal morphism in the root construction.

This proves the theorem.

COROLLARY 2.2. *Let* $\mathscr{A}' \subseteq \mathscr{A}$, *where both have zeros. If* $\mathscr{A}'$ *has pullbacks* (*intersections, inverse images, equalizers, images, limits, kernels, unions, adjoint* *situations, reflective theory) and* $\mathscr{A}$ *has pushouts (cointersections, coinverse images, coequalizers, coimages, colimits, cokernels, counions, co-adjoint situations, coreflective theory), then* $(\mathscr{A}', \mathscr{A})$ *has pullbacks (intersections, inverse images, equalizers, images, limits, kernels, unions, adjoint situations, reflective theory), respectively.*

THEOREM 2.3. *Hypothesis*: $\forall A \in \mathscr{A}\ \exists x \in [A', A]$, *where* $A' \in \mathscr{A}'$.
Conclusion.[3] *The converse of Theorem* 2.1 *holds.*

Proof. This is proved in Rine (1970) but is straightforward.

THEOREM 2.4. *Assume condition (H) and the hypothesis of Theorem* 2.1.
Conclusion: (1″) $(\mathscr{A}', \Delta, \mathscr{A})$ *can be extended to a category*;

(2″) $(\chi, \Delta, \chi^0) \xrightarrow{(F,\Delta,F^0)} (\mathscr{A}', \Delta, \mathscr{A})$ *has a left root* $\varinjlim(F, \Delta, F^0)$;

(3″) *in* $(\mathscr{A}', \Delta, \mathscr{A})$ $\varprojlim F \xrightarrow{\overleftarrow{\lim}(F,\Delta,F^0)} \varinjlim F^0$.

Moreover, one gets another converse.

THEOREM 2.5. *Assume condition (H) and the hypothesis of Theorem* 2.3.
Conclusion: *The converse of Theorem* 2.4 *holds.*

One can look at Theorems 2.4 and 2.5 in another way.
Let $(H, \mathscr{h}, h) \in [\mathfrak{A}, \mathscr{B}]$ and $(K, \mathscr{k}, k) \in [\mathscr{C}, \mathscr{B}]$, where $\operatorname{Im} \mathscr{h} = \operatorname{Im} \mathscr{k}$.

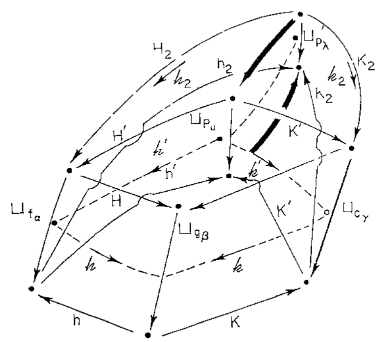


FIGURE 9

Without loss of generality (Rine, 1970), it is obvious that we may assume that $|\operatorname{Im} m_2|$[4] $> |\operatorname{Im} m_1|$, where $m_1 = i_A h$, $m_2 = i_C k$, i_A, i_C injections into

[3] Assume that coordinate maps can be considered unique w.r.t. (F, F^0) (e.g., with equalizers). Thus, γ_i', γ_j' s.t. (γ_i', γ_i), $(\gamma_j', \gamma_j) \in [(L, R)(m), (F, F^0)(m)]$ natural coordinate maps implies that $\gamma_i' = \gamma_j' \cdots$ 'reduction property'.

[4] Here, we are referring to cardinality (usually finite).

$A \times C$ (Rine, 1970). Hence, if P is the pushout object of h and k, then $|C| \geqslant |\operatorname{Im} m_2| \geqslant |P| > |\operatorname{Im} m_1|$. Moreover, since $\mathscr{S}$ (category of sets) has pullbacks, one can construct the pullback of $\hbar$ and $\&$. Now let P' be the pullback of H, K with maps H', K'; let h', k' be the pushout maps for P. We must first construct a well-defined family of maps $\cup p_\mu$, $p_\mu \in [P', P]$; then we construct appropriate $\hbar'$, $\&'$ such that $(H', \hbar', h') \in [(P', \cup p_\mu, P), \mathscr{A}]$ and $(K', \&', k') \in [(P', \cup p_\mu, P), \mathscr{C}]$. $\forall u \in P\, \exists p_\mu$ such that $p_\mu = h' f_\mu H'$ and $\hbar'(\mu) = \mu$; also, we have $p_\mu = k' c_\gamma K'$ and $\&(\mu) = \gamma$. $\forall \alpha\, \exists \gamma$ such that $p_\alpha = p_\gamma$. γ not unique, $p_\alpha = p_\gamma = p_{\gamma'} = \cdots$, since $p_\alpha = h' f_\alpha H' = h' h g_{\hbar(\alpha)} H H' = k' k g_{\hbar(\alpha)} K K' = k' k g_{\&(\gamma)} K K' = k' k g_{\&(\gamma')} K K' = k' c_\gamma K' = p_\gamma$ due to the property that $\operatorname{Im} \hbar = \operatorname{Im} \&$. Consider p_μ where $\cdots = p_{\alpha'} = p_\alpha = p_\mu = p_\gamma = p_{\gamma'} = \cdots$.

Finally, assume the existence of $(H_2, \hbar_2, h_2)$, $(K_2, \&_2, k_2)$ in

$$[(-, \cup p_\lambda', -), \mathscr{A}], \qquad [(-, \cup p_\lambda', -), \mathscr{C}],$$

respectively, such that $\mathscr{H}\mathscr{H}_2 = \mathscr{K}\mathscr{K}_2$. Then,

$$p_\lambda' = h_2 f_{\hbar_2(\lambda)} H_2 = h_2 h g_{\hbar(\hbar_2(\lambda))} H H_2 = k_2 k g_{\&(\&_2(\lambda))} K K_2 ;$$

or, $HH_2 = KK_2$, $h_2 h = k_2 k$, $\hbar\hbar_2 = \&\&_2$. Since $\mathscr{A}'$ has pullbacks, $\mathscr{A}$ has pushouts, and $\mathscr{S}$ has pullbacks, there exist unique maps into P' and out of P. We can now go to the general argument (Rine, 1970) in order to reach the conclusion of Theorem 2.4. An argument similar to that for Theorem 2.3 yields Theorem 2.5.

III. Conclusion

The power of Theorem 2.1 lies within the availability of *limit preserving functors from* S (Mitchell, 1965); for, as we shall see from Theorem 3.1, if P is some other system, every function b of $P = \cup \bar{b}$ has a *solution set in* S which acts like a set of replacements, i.e., what jobs can b *do* in S. In figure 10, C_i is a component (maximal connected diagram), $S_j(b)$ is a member of b's solution set in S, and a_k is the limit of component C_k.

Let $\mathscr{A}' \subseteq \mathscr{A}$, where $\mathscr{A}'$ has limits and $\mathscr{A}$ has colimits. Let χ_1, χ_2 be connected schemes, and let $\mathscr{B}' \subseteq \mathscr{B}$ be a subcategory of $\mathscr{B}$. Suppose that $\mathscr{B}'$ has limits and $\mathscr{B}$ has colimits. Let G_1, G_2 be functors from χ_1 to $\mathscr{A}'$ and χ_2 to $\mathscr{B}'$, respectively. By Theorem, there exist machines $\varprojlim(G_1, G_1^0) \in [\varprojlim G_1, \varprojlim G_1^0]$, $\varprojlim(G_2, G_2^0) \in [\varprojlim G_2, \varprojlim G_2^0]$. Thus, the following theorem by Mitchell (1965) is quite useful.

Theorem (Mitchell) 3.1. *Consider a covariant functor $T : \mathscr{C} \to \mathscr{D}$ where $\mathscr{C}$*

is complete and locally small. Then T has a coadjoint if and only if it is a limit preserving functor which admits a solution set for every object in $\mathscr{D}$.

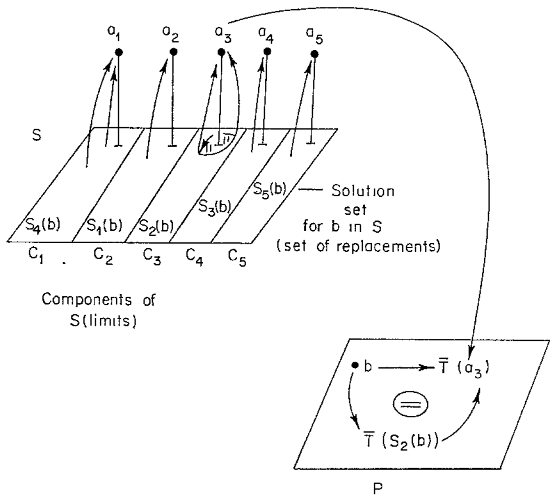


FIGURE 10

Another way of representing replacement of input objects is illustrated in Fig. 11. A *second level replacement* is pictured.

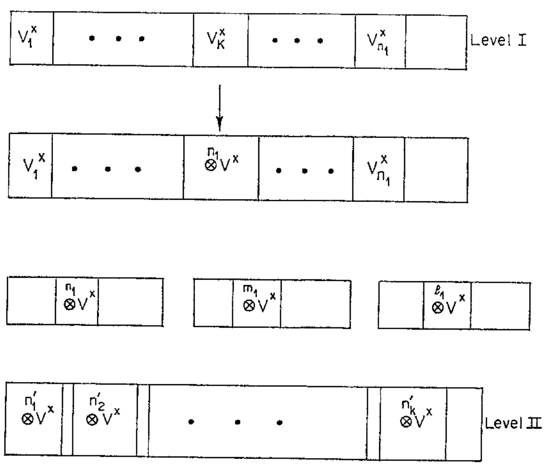


FIGURE 11

• • • and the story repeats itself.

RECEIVED: September 1, 1970; REVISED: February 11, 1971

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